

Subfactor categories of triangulated categories

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Abstract

Let \mathcal{T} be a triangulated category, \mathcal{A} a full subcategory of \mathcal{T} and \mathcal{X} a functorially finite subcategory of \mathcal{A} . If \mathcal{A} has the properties that any \mathcal{X} -monomorphism of \mathcal{A} has a cone and any \mathcal{X} -epimorphism has a cocone. Then the subfactor category $\mathcal{A}/[\mathcal{X}]$ admits a pretriangulated structure in the sense of [BR]. Moreover the above pretriangulated category $\mathcal{A}/[\mathcal{X}]$ with $(\mathcal{X}, \mathcal{X}[1]) = 0$ becomes a triangulated category if and only if $(\mathcal{A}, \mathcal{A})$ forms an \mathcal{X} -mutation pair and \mathcal{A} is closed under extensions.

Key words: triangulated category; right triangulated category; subfactor triangulated category; mutation pair

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1 Introduction

Over the past decades, triangulated categories have made their way into many different parts of mathematics and have become indispensable in many different areas of mathematics. Nowadays there are important applications of triangulated categories in areas like algebraic geometry, algebraic topology, commutative algebra, differential geometry, microlocal analysis or representation theory.

The most influenced work of triangulated categories in representation theory was created by Happel [H], he proved that the stable category of a Frobenius category is a triangulated category in last two decades. Later Beligiannis, Marmaridis and

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Reiten et al. studied the one-side triangulated categories in a series of works [ABM, BM, BR, B], one of the importance results of them is that any contravariantly (resp. covariantly) finite subcategory \mathcal{X} of $\text{mod}\Lambda$ induces on the stable category $\underline{\text{mod}}_{\mathcal{X}}\Lambda$ (resp. $\overline{\text{mod}}_{\mathcal{X}}\Lambda$) of $\text{mod}\Lambda$ a left (resp. right) triangulated category, where $\text{mod}\Lambda$ is the category of finitely generated Λ -modules over an artin algebra Λ . In the recent paper [IY], Iyama and Yoshino proved that if $\mathcal{D} \subset \mathcal{Z}$ are extension closed subcategories of a triangulated category \mathcal{T} with \mathcal{D} satisfying $(\mathcal{D}, \mathcal{D}[1])=0$, and if $(\mathcal{Z}, \mathcal{Z})$ is a \mathcal{D} -mutation pair. Then the subfactor category $\mathcal{Z}/[\mathcal{D}]$ called by Iyama and Yoshino is a triangulated category. Soon later, Jørgensen [J] gave a similar construction of triangulated category by quotient category in another manner, and Liu and Zhu[LZ] applied these constructions to the one-side triangulated categories.

In this paper, we construct a one-side triangulated structure for the subfactor category $\mathcal{T}/[\mathcal{X}]$ where \mathcal{X} is a contravariantly or covariantly finite subcategory of a triangulated category \mathcal{T} .

Main Theorem 2.9.

Let \mathcal{A} be a full subcategory of a triangulated category \mathcal{T} , \mathcal{X} a covariantly finite subcategory and \mathcal{Y} a contravariantly finite subcategory of \mathcal{A}

1. If the subcategory \mathcal{A} has the property that any \mathcal{X} -monomorphism of \mathcal{A} has a cone, then the subfactor category $\mathcal{A}/[\mathcal{X}]$ forms a right triangulated category.
2. If the subcategory \mathcal{A} has the property that any \mathcal{Y} -epimorphism of \mathcal{A} has a cocone, then the subfactor category $\mathcal{A}/[\mathcal{Y}]$ forms a left triangulated category.

Moreover, If \mathcal{X} is a functorially finite subcategory of \mathcal{A} and \mathcal{A} has the properties that any \mathcal{X} -monomorphism of \mathcal{A} has a cone and any \mathcal{X} -epimorphism has a cocone, then the subfactor category $\mathcal{A}/[\mathcal{X}]$ forms a pretriangulated category. Noting that when the subcategory \mathcal{A} is \mathcal{T} , the theorem is just the Theorem 1.2 of [J]. Later, we apply our results to the \mathcal{D} -mutation setting of [IY], and get the following result: The above pretriangulated category $\mathcal{A}/[\mathcal{X}]$ with $(\mathcal{X}, \mathcal{X}[1]) = 0$ becomes a triangulated category if and only if $(\mathcal{A}, \mathcal{A})$ forms an \mathcal{X} -mutation pair and \mathcal{A} is closed under extensions.

2 Subfactor categories

Throughout this paper we assume, unless other stated, that all consider categories are k -linear Hom-finite, skeletally small, and Krull-Schmidt, where k is a field. We denote by $\text{Hom}_{\mathcal{C}}(X, Y)$ or $\mathcal{C}(X, Y)$ the set of morphisms from $X \rightarrow Y$ in a category \mathcal{C} . When we say that \mathcal{D} is a subcategory of \mathcal{C} , we always mean that \mathcal{D} is a full subcategory which is closed under isomorphisms, direct sums and direct summands.

We begin by recall some definitions and notations of approximations and homologically finite subcategory of an arbitrary category[AR]. More information please refer to [AR].

Let \mathcal{A} be a category and \mathcal{X} a subcategory of \mathcal{A} . A morphism $f : X_B \rightarrow B$ of \mathcal{A} with X_B an object in \mathcal{X} , is said to be a *right \mathcal{X} -approximation* of B , if the morphism $\mathcal{A}(X, f_B) : \mathcal{A}(X, X_B) \rightarrow \mathcal{A}(X, B)$ is surjective for all objects X in \mathcal{X} . The subcategory \mathcal{X} is said to be a *contravariantly finite subcategory* of \mathcal{A} if any object B of \mathcal{A} has a right \mathcal{X} -approximation. Dually a *left \mathcal{X} -approximation* and a *covariantly finite subcategory* of \mathcal{A} are defined. A contravariantly and covariantly finite subcategory is called *functorially finite*.

Given two objects A and B of \mathcal{A} , we denote by $[\mathcal{X}](A, B)$ the set of morphisms from A to B of \mathcal{A} which factor through some object of \mathcal{X} . It is well known that $[\mathcal{X}](A, B)$ is a subgroup of $\mathcal{A}(A, B)$, and that the family of these subgroups $[\mathcal{X}](A, B)$ forms an ideal of \mathcal{A} . Thus we have the category $\mathcal{A}/[\mathcal{X}]$ whose objects are objects of \mathcal{A} and whose morphisms are elements of $\mathcal{A}(A, B)/[\mathcal{X}](A, B)$. The composition of $\mathcal{A}/[\mathcal{X}]$ is induced canonically by the composition of \mathcal{A} . We denote by \bar{A} the image of an object A of \mathcal{A} in $\mathcal{A}/[\mathcal{X}]$ and \bar{f} the image of $f : A \rightarrow B$ of \mathcal{A} in $\mathcal{A}/[\mathcal{X}]$.

For basic references on representation theory of triangulated categories, we refer to [H].

We recall some basic notions on one-sided triangulated categories from [ABM, BM, BR]. Let \mathcal{C} be an additive category and $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ an additive endofunctor called *suspension functor*. A *sextuple* (A, B, C, f, g, h) in \mathcal{C} is given by the form of $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ with $A, B, C \in \mathcal{C}$. A *morphism from sextuples* (A, B, C, f, g, h) to (A', B', C', f', g', h') is a triple (α, β, γ) of morphisms of \mathcal{C} , which makes the next diagram commutative:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \Sigma \alpha \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A' \end{array}$$

If in addition α, β , and γ are isomorphisms in \mathcal{C} , the morphism (α, β, γ) is then called an *isomorphism of sextuples*.

The composition of the morphisms of sextuples is induced in the canonical way by the corresponding composition of the morphisms of \mathcal{C} .

Definition 2.1. A set ∇ of sextuples in \mathcal{C} is called a *right triangulation* of \mathcal{C} if it is closed under isomorphisms and satisfies the following axioms. The elements of ∇ are then called *right triangles*.

[rTR0] For any object A of \mathcal{C} , the sextuple $0 \xrightarrow{0} A \xrightarrow{1_A} A \xrightarrow{0} 0$ belongs to ∇

[rTR1] Every morphism $f : A \rightarrow B$ in \mathcal{C} can be embedded into a right triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$.

[rTR2] If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ is a right triangle, then $B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{-\Sigma f} \Sigma B$ is a right triangle.

[rTR3] Given two right triangles $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ and $A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A'$

and morphisms $\alpha : A \rightarrow A'$, $\beta : B \rightarrow B'$ such that $\beta f = f' \alpha$, there exists a morphism (α, β, γ) from the first triangle to the second.

[rTR4] Given right triangles $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$, $B \xrightarrow{a} X \xrightarrow{b} Y \xrightarrow{c} \Sigma B$ and $A \xrightarrow{af} X \xrightarrow{d} Z \xrightarrow{e} \Sigma A$. Then there exist morphism $s : C \rightarrow Z$ and $t : Z \rightarrow Y$ such that the following diagrams commute and the third column in first diagram is a right triangle.

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\
\parallel & & \downarrow a & & \downarrow s & & \parallel \\
A & \xrightarrow{af} & X & \xrightarrow{d} & Z & \xrightarrow{e} & \Sigma A \\
& & \downarrow b & & \downarrow t & & \\
& & Y & \xlongequal{\quad} & Y & & \\
& & \downarrow c & & \downarrow & & \\
& & \Sigma B & \longrightarrow & \Sigma C & &
\end{array}$$

$$\begin{array}{ccccccc}
A & \xrightarrow{af} & X & \xrightarrow{d} & Z & \xrightarrow{e} & \Sigma A \\
\downarrow f & & \downarrow 1 & & \downarrow t & & \downarrow \Sigma f \\
B & \xrightarrow{a} & X & \xrightarrow{b} & Y & \xrightarrow{c} & \Sigma B
\end{array}$$

The additive category \mathcal{C} together with the suspension functor Σ and the right triangulation ∇ is called a *right triangulated category*, denoted by the triple $(\mathcal{C}, \Sigma, \nabla)$.

The left triangulated category $(\mathcal{C}, \Omega, \Delta)$ is defined dually, where the endofunctor $\Omega : \mathcal{C} \rightarrow \mathcal{C}$ is called *loop functor* and Δ is the set of left triangles satisfying axioms analogous to rTR0-rTR4.

Let \mathcal{T} be a triangulated category and \mathcal{X} a subcategory of \mathcal{T} . We recall that a morphism $f : A \rightarrow B$ is called \mathcal{X} -*monic*, if the induced morphism $\mathcal{T}(f, X) : \mathcal{T}(B, X) \rightarrow \mathcal{T}(A, X)$ is surjective for any object X of \mathcal{X} . Dually a \mathcal{X} -*epimorphism* morphism is defined. Obviously, any left \mathcal{X} -approximation is \mathcal{X} -monic and any right \mathcal{X} -approximation is \mathcal{X} -epic.

Lemma 2.2. *Let $l : A \rightarrow X_A$ be a \mathcal{X} -monomorphism and $f : A \rightarrow B$ any morphism in \mathcal{T} . Then*

- (1) *The morphism $\begin{pmatrix} f \\ l \end{pmatrix} : A \rightarrow B \oplus X_A$ is also \mathcal{X} -monic.*
- (2) *The morphism g in the following commutative diagram with triangles for rows is also \mathcal{X} -monic.*

$$\begin{array}{ccccccc}
A'[-1] & \longrightarrow & A & \xrightarrow{l} & X_A & \longrightarrow & A' \\
\parallel & & \downarrow f & & & & \parallel \\
A'[-1] & \longrightarrow & B & \xrightarrow{g} & C & \longrightarrow & A'
\end{array}$$

Proof. (1) It is trivial.

(2) By Lemma 1.4.3 in [N], the diagram may be completed to a morphism of triangles

$$\begin{array}{ccccccc} A'[-1] & \longrightarrow & A & \xrightarrow{l} & X_A & \longrightarrow & A' \\ \parallel & & \downarrow f & & \downarrow k & & \parallel \\ A'[-1] & \longrightarrow & B & \xrightarrow{g} & C & \longrightarrow & A' \end{array}$$

such that

$$\begin{array}{ccc} A & \xrightarrow{l} & X_A \\ \downarrow f & & \downarrow k \\ B & \xrightarrow{g} & C \end{array}$$

is homotopy cartesian. Since l is \mathcal{X} -monic, for any $\varphi : B \rightarrow X$, $\forall X \in \mathcal{X}$ there exists a morphism $\psi : X_A \rightarrow X$ such that $\psi l = \varphi f$. Hence there exists a morphism $\sigma : C \rightarrow X$ such that $\varphi = \sigma g$ by the property of homotopy cartesian, i.e. g is \mathcal{X} -monic. \square

For the rest of the paper, we shall deal only with the right case, leaving for the reader to state and prove dual results for the left case. Let \mathcal{T} be a triangulated category and $\mathcal{X} \subset \mathcal{A}$ full subcategories of \mathcal{T} . From now on, we assume, unless otherwise stated, that \mathcal{A} and \mathcal{X} satisfy the following two conditions:

(A1) \mathcal{X} is a covariantly finite subcategory of \mathcal{A} .

(A2) Any \mathcal{X} -monomorphism of \mathcal{A} has a cone. i.e. for any \mathcal{X} -monomorphism $f : A \rightarrow B$ of \mathcal{A} , the third term C_f in the triangle $A \xrightarrow{f} B \rightarrow C_f \rightarrow A[1]$ of \mathcal{T} is also in \mathcal{A} .

Under such a setting, we will enrich the *subfactor category* $\mathcal{A}/[\mathcal{X}]$ of \mathcal{A} a right triangulated structure.

In order to construct the right triangulation ∇ of $\mathcal{A}/[\mathcal{X}]$, First of all, we construct the *suspension functor* $\Sigma : \mathcal{A}/[\mathcal{X}] \rightarrow \mathcal{A}/[\mathcal{X}]$ as follows: For any object $A \in \mathcal{A}$, consider the triangle in \mathcal{T}

$$A \xrightarrow{\alpha_A} X_A \longrightarrow \Sigma A \longrightarrow A[1]$$

where α_A is a left \mathcal{X} -approximation of A and define $\Sigma \bar{A}$ like this. For any morphism $f \in (A, A')$, there exist g and h which make the following diagram commutative.

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha_A} & X_A & \longrightarrow & \Sigma A & \longrightarrow & A[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ A' & \xrightarrow{\alpha_{A'}} & X_{A'} & \longrightarrow & \Sigma A' & \longrightarrow & A'[1] \end{array}$$

Now put $\Sigma \bar{f} := \bar{h}$ and it is easy to see that \bar{h} is uniquely determined by \bar{f} , i.e. the endofunctor $\Sigma : \mathcal{A}/[\mathcal{X}] \rightarrow \mathcal{A}/[\mathcal{X}]$ is well defined.

Next we construct two kinds of right triangles in $\mathcal{A}/[\mathcal{X}]$, the distinguished and the induced ones.

The distinguished right triangles are obtained as follows:

Given a morphism $f : A \rightarrow B$ in \mathcal{A} , consider the following diagram with triangles for rows in \mathcal{T}

$$\begin{array}{ccccccc} \Sigma A[-1] & \longrightarrow & A & \xrightarrow{\alpha_A} & X_A & \longrightarrow & \Sigma A \\ & & \downarrow f & & \downarrow k & & \parallel \\ \Sigma A[-1] & \longrightarrow & B & \xrightarrow{g} & C_f & \xrightarrow{h} & \Sigma A \end{array}$$

where α_A is a left \mathcal{X} -approximation of A . Noting that by Lemma 2.2 the morphism $\begin{pmatrix} f \\ \alpha_A \end{pmatrix}$ is \mathcal{X} -monic and there exists a morphism k such that

$$\begin{array}{ccc} A & \xrightarrow{\alpha_A} & X_A \\ \downarrow f & & \downarrow k \\ B & \xrightarrow{g} & C_f \end{array}$$

is homotopy cartesian, i.e. $A \xrightarrow{\begin{pmatrix} f \\ \alpha_A \end{pmatrix}} B \oplus X_A \xrightarrow{(-g, k)} C_f \longrightarrow A[1]$ is a triangle in \mathcal{T} , hence C_f is in \mathcal{A} .

Definition 2.3. A sextuple $\overline{X} \xrightarrow{\overline{u}} \overline{Y} \xrightarrow{\overline{v}} \overline{Z} \xrightarrow{\overline{w}} \Sigma \overline{X}$ in $\mathcal{A}/[\mathcal{X}]$ is said to be an \mathcal{X} -*distinguished* right triangle, if it is isomorphic to a sextuple $\overline{A} \xrightarrow{\overline{f}} \overline{B} \xrightarrow{\overline{g}} \overline{C_f} \xrightarrow{\overline{h}} \Sigma \overline{A}$ given by some $f : A \rightarrow B$ in \mathcal{A} .

We shall call the \mathcal{X} -distinguished right triangle just distinguished right triangle whenever it is clear from the context.

Remark 2.4. Noting that the sextuple $\overline{A} \xrightarrow{\overline{f}} \overline{B} \xrightarrow{\overline{g}} \overline{C_f} \xrightarrow{\overline{h}} \Sigma \overline{A}$ is independent of choice of k and is uniquely determined (up to isomorphism) by f . We can assume, without loss of generality, that the sextuple is given by the following commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\alpha_A} & X_A & \longrightarrow & \Sigma A \\ \downarrow f & & \downarrow k & & \parallel \\ B & \xrightarrow{g} & C_f & \xrightarrow{h} & \Sigma A \end{array}$$

with triangles for rows and the left square being homotopy cartesian.

The induced right triangles are obtained as follows:

Consider the triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h'} A[1]$ in \mathcal{T} with $f \in \mathcal{A}$ being \mathcal{X} -monic,

there is a commutative diagram of triangles:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h'} & A[1] \\ \parallel & & \downarrow & & \downarrow h & & \parallel \\ A & \xrightarrow{\alpha_A} & X_A & \xrightarrow{\beta_A} & \Sigma A & \xrightarrow{\gamma_A} & A[1] \end{array}$$

where α_A is a left \mathcal{X} -approximation of A .

Remark 2.5. Noting that the morphism \bar{h} in $\mathcal{A}/[\mathcal{X}]$ is unique determined by the triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h'} A[1]$. Indeed, if $a : C \rightarrow \Sigma A$ satisfies $h' = \gamma l$, then $\gamma h = \gamma l$, i.e. $\gamma(h - l) = 0$. Hence $h - l$ factor through X_A , i.e. $\bar{h} = \bar{l}$.

Definition 2.6. A sextuple $\bar{X} \xrightarrow{\bar{u}} \bar{Y} \xrightarrow{\bar{v}} \bar{Z} \xrightarrow{\bar{w}} \Sigma \bar{X}$ in $\mathcal{A}/[\mathcal{X}]$ is said to be an \mathcal{X} -induced right triangle, if it is isomorphic to a sextuple $\bar{A} \xrightarrow{\bar{f}} \bar{B} \xrightarrow{\bar{g}} \bar{C} \xrightarrow{-\bar{h}} \Sigma \bar{A}$ for some \mathcal{X} -monomorphism $f : A \rightarrow B$ of \mathcal{A} .

We shall call the \mathcal{X} -induced right triangle just induced right triangle whenever it is clear from the context.

In fact, the set ∇_d of right distinguished triangles and the set ∇_i of right induced triangles are equal.

Proposition 2.7. Any distinguished right triangle is isomorphic to an induced one and any induced right triangle is isomorphic to a distinguished one in $\mathcal{A}/[\mathcal{X}]$.

Proof. Given a morphism $f : A \rightarrow B$ in \mathcal{A} , by Remark 2.4, the right distinguished triangle $\bar{A} \xrightarrow{\bar{f}} \bar{B} \xrightarrow{\bar{g}} \bar{C}_f \xrightarrow{\bar{h}} \Sigma \bar{A}$ is given by the following commutative diagram of triangles

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha_A} & X_A & \xrightarrow{\beta_A} & \Sigma A & \xrightarrow{\gamma_A} & A[1] \\ \downarrow f & & \downarrow k & & \parallel & & \downarrow f[1] \\ B & \xrightarrow{g} & C_f & \xrightarrow{h} & \Sigma A & \longrightarrow & A[1] \end{array}$$

with the left square being homotopy cartesian, that is to say there is a triangle of \mathcal{T}

$$A \xrightarrow{\begin{pmatrix} f \\ \alpha_A \end{pmatrix}} B \oplus X_A \xrightarrow{(-g, k)} C_f \xrightarrow{\gamma_A h} A[1]$$

with $A, B \oplus X_A, C_f \in \mathcal{A}$. Since $\begin{pmatrix} f \\ \alpha_A \end{pmatrix}$ is \mathcal{X} -monic, the triangle induces a right induced triangle in $\mathcal{A}/[\mathcal{X}]$ as follows:

$$\begin{array}{ccccccc} A & \xrightarrow{\begin{pmatrix} f \\ \alpha_A \end{pmatrix}} & B \oplus X_A & \xrightarrow{(-g, k)} & C_f & \xrightarrow{\gamma_A h} & A[1] \\ \parallel & & \downarrow (0, 1) & & \downarrow h & & \parallel \\ A & \xrightarrow{\alpha_A} & X_A & \xrightarrow{\beta_A} & \Sigma A & \xrightarrow{\gamma_A} & A[1] \end{array}$$

Hence the induced triangle is $\bar{A} \xrightarrow{\bar{f}} \bar{B} \xrightarrow{-\bar{g}} \bar{C}_f \xrightarrow{-\bar{h}} \Sigma \bar{A}$, obviously isomorphic to the distinguished one $\bar{A} \xrightarrow{\bar{f}} \bar{B} \xrightarrow{\bar{g}} \bar{C}_f \xrightarrow{\bar{h}} \Sigma \bar{A}$.

Given a right induced triangle $\bar{A} \xrightarrow{\bar{f}} \bar{B} \xrightarrow{\bar{g}} \bar{C} \xrightarrow{-\bar{\theta}} \Sigma \bar{A}$, i.e. we have the following commutative diagram of triangles

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & A[1] \\ \parallel & & \downarrow \delta & & \downarrow \theta & & \parallel \\ A & \xrightarrow{\alpha_A} & X_A & \xrightarrow{\beta_A} & \Sigma A & \xrightarrow{\gamma_A} & A[1] \end{array}$$

with $f, \alpha_A \in \mathcal{A}$ being \mathcal{X} -monic. By Remark 2.5, we can assume without loss of generality that $\theta : C \rightarrow \Sigma$ is the morphism such that the middle square is homotopy cartesian, then we have a triangle

$$B \xrightarrow{\begin{pmatrix} g \\ \delta \end{pmatrix}} C \oplus X_A \xrightarrow{(\theta, -\beta_A)} \Sigma A \xrightarrow{-f[1]\gamma_A} A[1]$$

An easy computation allow us to have the following commutative diagram of triangles.

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha_A} & X_A & \xrightarrow{\beta_A} & \Sigma A & \xrightarrow{\gamma_A} & A[1] \\ \downarrow -f & & \downarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix} & & \parallel & & \downarrow -f[1] \\ B & \xrightarrow{\begin{pmatrix} g \\ \delta \end{pmatrix}} & C \oplus X_A & \xrightarrow{(\theta, -\beta)} & \Sigma A & \xrightarrow{-f[1]\gamma_A} & B[1] \end{array}$$

Then by the definition of right distinguished triangles, the sextuple $\bar{A} \xrightarrow{\bar{f}} \bar{B} \xrightarrow{\bar{g}} \bar{C} \xrightarrow{-\bar{\theta}} \Sigma \bar{A}$ is the right distinguished triangle given by $-f$, obviously isomorphic to the induced triangle $\bar{A} \xrightarrow{\bar{f}} \bar{B} \xrightarrow{\bar{g}} \bar{C} \xrightarrow{-\bar{\theta}} \Sigma \bar{A}$. \square

In order to prove our main theorem, we also need the following lemma.

Lemma 2.8. *Any commutative diagram*

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{e} & A[1] \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow a[1] \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{e'} & A'[1] \end{array}$$

of triangles in \mathcal{T} with \mathcal{X} -monomorphisms f, f' of \mathcal{A} induces a commutative diagram

$$\begin{array}{ccccccc} \bar{A} & \xrightarrow{\bar{f}} & \bar{B} & \xrightarrow{\bar{g}} & \bar{C} & \xrightarrow{\bar{h}} & \Sigma \bar{A} \\ \downarrow \bar{a} & & \downarrow \bar{b} & & \downarrow \bar{c} & & \downarrow \Sigma \bar{a} \\ \bar{A}' & \xrightarrow{\bar{f}'} & \bar{B}' & \xrightarrow{\bar{g}'} & \bar{C}' & \xrightarrow{\bar{h}'} & \Sigma \bar{A}' \end{array}$$

of right induced triangles in $\mathcal{A}/[\mathcal{X}]$.

Proof. We have the following three commutative diagrams

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{e} & A[1] \\ \parallel & & \downarrow & & \downarrow -h & & \parallel \\ A & \xrightarrow{\alpha_A} & X_A & \xrightarrow{\beta_A} & \Sigma A & \xrightarrow{\gamma_A} & A[1] \end{array}$$

$$\begin{array}{ccccccc} A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{e'} & A'[1] \\ \parallel & & \downarrow & & \downarrow -h' & & \parallel \\ A' & \xrightarrow{\alpha'_A} & X'_A & \xrightarrow{\beta'_A} & \Sigma A' & \xrightarrow{\gamma'_A} & A[1] \end{array}$$

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha_A} & X_A & \xrightarrow{\beta_A} & \Sigma A & \xrightarrow{\gamma_A} & A[1] \\ \downarrow a & & \downarrow & & \downarrow a' & & \downarrow a[1] \\ A' & \xrightarrow{\alpha'_A} & X'_A & \xrightarrow{\beta'_A} & \Sigma A' & \xrightarrow{\gamma'_A} & A[1]' \end{array}$$

By the definition of suspension functor Σ , we have $\Sigma \bar{a} = \bar{a}'$ and $\gamma_A \cdot (-h) = e$, $\gamma_A \cdot (-h') = e'$. Then

$$\gamma'_A(a'h - h'c) = \gamma'_A a'h - \gamma'_A h'c = a[1]\gamma_A h - e'c = e'c - a[1]e = 0.$$

Hence $a'h - h'c$ factor through \mathcal{X} , i.e. $\Sigma \bar{a}h = \bar{h}'\bar{c}$. \square

Theorem 2.9. *Let \mathcal{A} be a full subcategory of a triangulated category \mathcal{T} , \mathcal{X} a covariantly finite subcategory and \mathcal{Y} a contravariantly finite subcategory of \mathcal{A}*

- (1). *If the subcategory \mathcal{A} has the property that any \mathcal{X} -monomorphism of \mathcal{A} has a cone, then the subfactor category $\mathcal{A}/[\mathcal{X}]$ forms a right triangulated category.*
- (2). *If the subcategory \mathcal{A} has the property that any \mathcal{Y} -epimorphism of \mathcal{A} has a cocone, then the subfactor category $\mathcal{A}/[\mathcal{Y}]$ forms a left triangulated category.*

Proof. (1) Let ∇ be the set of the right distinguished triangles of $\mathcal{A}/[\mathcal{X}]$. We only need to prove the set ∇ is a right triangulation of $\mathcal{A}/[\mathcal{X}]$. It is sufficient to check the axioms (rTR0) to (rTR4) in Definition 2.1.

(rTR0) The sextuple $\bar{A} \xrightarrow{\bar{id}} \bar{A} \xrightarrow{0} 0 \xrightarrow{0} \Sigma \bar{A}$ is obviously a right distinguished triangle of $\mathcal{A}/[\mathcal{X}]$ given by the identity $id : A \rightarrow A$, since we have the following diagram of triangles.

$$\begin{array}{ccccc} A & \xrightarrow{\alpha_A} & X_A & \longrightarrow & \Sigma A \\ \downarrow id & & \downarrow id & & \parallel \\ A & \xrightarrow{\alpha_A} & X_A & \longrightarrow & \Sigma A \end{array}$$

(rTR1) Any morphism $\bar{f} : A \rightarrow B$ in $\mathcal{A}/[\mathcal{X}]$ can be embedded into a right distinguished triangle $\bar{A} \xrightarrow{\bar{f}} \bar{B} \xrightarrow{\bar{g}} \bar{C}_f \xrightarrow{\bar{h}} \Sigma \bar{A}$ of $\mathcal{A}/[\mathcal{X}]$ since we have the following commutative diagram of triangles in \mathcal{T}

$$\begin{array}{ccccc} A & \xrightarrow{\alpha_A} & X_A & \longrightarrow & \Sigma A \\ \downarrow f & & \downarrow k & & \parallel \\ B & \xrightarrow{g} & C_f & \xrightarrow{h} & \Sigma A \end{array}$$

with $A, B, C_f, \Sigma A \in \mathcal{A}$.

(rTR2) Let $\bar{A} \xrightarrow{\bar{f}} \bar{B} \xrightarrow{\bar{g}} \bar{C}_f \xrightarrow{\bar{h}} \Sigma \bar{A}$ be a right distinguished triangle given by $f \in \mathcal{A}$, i.e. there is the following commutative diagram of triangles of \mathcal{T} .

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha_A} & X_A & \longrightarrow & \Sigma A & \xrightarrow{\gamma_A} & A[1] \\ \downarrow f & & \downarrow k & & \parallel & & \downarrow f[1] \\ B & \xrightarrow{g} & C_f & \xrightarrow{h} & \Sigma A & \xrightarrow{f[1]\gamma_A} & B[1] \end{array} \quad (1)$$

Since α_A is \mathcal{X} -monic, the morphism $g \in \mathcal{A}$ is also \mathcal{X} -monic by Lemma 2.2. Consider the following commutative diagram

$$\begin{array}{ccccccc} B & \xrightarrow{g} & C_f & \xrightarrow{h} & \Sigma A & \xrightarrow{f[1]\gamma_A} & B[1] \\ \parallel & & \downarrow & & \downarrow u & & \parallel \\ B & \xrightarrow{\alpha_B} & X_B & \xrightarrow{\beta_B} & \Sigma B & \xrightarrow{\gamma_B} & B[1] \end{array} \quad (2)$$

of triangles in \mathcal{T} , i.e. the triangle $B \xrightarrow{g} C_f \xrightarrow{h} \Sigma A \xrightarrow{f[1]\gamma_A} B[1]$ of \mathcal{T} induces the right induced triangle $\bar{B} \xrightarrow{\bar{g}} \bar{C}_f \xrightarrow{\bar{h}} \Sigma \bar{A} \xrightarrow{-\bar{u}} \Sigma \bar{B}$ of $\mathcal{A}/[\mathcal{X}]$. By composing the commutative diagrams (1) and (2), we have the following commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha_A} & X_A & \xrightarrow{\beta_A} & \Sigma A & \xrightarrow{\gamma_A} & A[1] \\ \downarrow f & & \downarrow & & \downarrow u & & \downarrow f[1] \\ B & \xrightarrow{\alpha_B} & X_B & \xrightarrow{\beta_B} & \Sigma B & \xrightarrow{\gamma_B} & B[1] \end{array}$$

of triangles, then $\Sigma \bar{f} = \bar{u}$ by the construction of suspension functor Σ in $\mathcal{A}/[\mathcal{X}]$. Thus the right induced triangle $\bar{B} \xrightarrow{\bar{g}} \bar{C}_f \xrightarrow{\bar{h}} \Sigma \bar{A} \xrightarrow{-\Sigma \bar{f}} \Sigma \bar{B}$ belongs to ∇ by Proposition 2.7.

(rTR3) Take a commutative diagram

$$\begin{array}{ccccccc} \bar{A} & \xrightarrow{\bar{f}} & \bar{B} & \xrightarrow{\bar{g}} & \bar{C} & \xrightarrow{\bar{h}} & \Sigma \bar{A} \\ \downarrow \bar{a} & & \downarrow \bar{b} & & & & \downarrow \Sigma \bar{a} \\ \bar{A}' & \xrightarrow{\bar{f}'} & \bar{B}' & \xrightarrow{\bar{g}'} & \bar{C}' & \xrightarrow{\bar{h}'} & \Sigma \bar{A}' \end{array}$$

of right distinguished triangles in $\mathcal{A}/[\mathcal{X}]$. Owing to Proposition 2.7, we can assume that these triangles are induced, i.e. $\bar{A} \xrightarrow{\bar{f}} \bar{B} \xrightarrow{\bar{g}} \bar{C} \xrightarrow{\bar{h}} \Sigma \bar{A}$ and $\bar{A}' \xrightarrow{\bar{f}'} \bar{B}' \xrightarrow{\bar{g}'} \bar{C}' \xrightarrow{\bar{h}'} \Sigma \bar{A}'$ are induced by the triangles $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{e} A[1]$ and $A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{e'} A'[1]$ with $f, f' \in \mathcal{A}$ being \mathcal{X} -monic, respectively. By the construction of right induced triangles, we have the following (not necessarily commutative) diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{e} & A[1] \\ \downarrow a & & \downarrow b & & & & \downarrow a[1] \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{e'} & A'[1] \end{array}$$

of triangles in \mathcal{T} . Since $\bar{b}\bar{f} = \bar{f}'\bar{a}$ holds, $bf - f'a$ factors through \mathcal{X} , i.e. there exist morphisms $t : A \rightarrow X$ and $t' : X \rightarrow Y$ for some $X \in \mathcal{X}$ such that $bf - f'a = t't$. Because f is \mathcal{X} -monic, there exists a morphism $s : B \rightarrow X$ such that $t = sf$. Then $bf - f'a = (t's)f$ with $t's \in [\mathcal{X}](B, B')$. Set $b' = b - t's$, then we have $b'f = bf - t'sf = f'a$. Then we get the following commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{e} & A[1] \\ \downarrow a & & \downarrow b' & & \downarrow \exists c & & \downarrow a[1] \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{e'} & A'[1] \end{array}$$

of triangles in \mathcal{T} with $f, f' \in \mathcal{A}$ being \mathcal{X} -monic. Thus the assertion follows from Lemma 2.8, since $\bar{b}' = \bar{b}$.

(rTR4) As the argumentation in (rTR3), we can assume by Proposition 2.7, the triangles are induced, i.e. $\bar{A} \xrightarrow{\bar{f}} \bar{B} \xrightarrow{\bar{g}} \bar{C} \xrightarrow{\bar{h}} \Sigma \bar{A}$, $\bar{B} \xrightarrow{\bar{a}} \bar{X} \xrightarrow{\bar{b}} \bar{Y} \xrightarrow{\bar{c}} \Sigma \bar{B}$ and $\bar{A} \xrightarrow{\bar{a}\bar{f}} \bar{X} \xrightarrow{\bar{d}} \bar{Z} \xrightarrow{\bar{e}} \Sigma \bar{A}$ are induced by the triangles $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h'} A[1]$, $B \xrightarrow{a} X \xrightarrow{b} Y \xrightarrow{c'} B[1]$ and $A \xrightarrow{af} X \xrightarrow{d} Z \xrightarrow{e'} A[1]$ with $f, a, af \in \mathcal{A}$ being \mathcal{X} -monic. Then by the octahedral axioms in \mathcal{T} , we have the following commutative

diagrams

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h'} & A[1] \\
\parallel & & \downarrow a & & \downarrow s & & \parallel \\
A & \xrightarrow{af} & X & \xrightarrow{d} & Z & \xrightarrow{e'} & A[1] \\
& & \downarrow b & & \downarrow t & & \\
& & Y & \xlongequal{\quad} & Y & & \\
& & \downarrow c' & & \downarrow & & \\
& & B[1] & \longrightarrow & C[1] & &
\end{array}$$

$$\begin{array}{ccccccc}
A & \xrightarrow{af} & X & \xrightarrow{d} & Z & \xrightarrow{e'} & A[1] \\
\downarrow f & & \parallel & & \downarrow t & & \downarrow f[1] \\
B & \xrightarrow{a} & X & \xrightarrow{b} & Y & \xrightarrow{c'} & B[1]
\end{array}$$

of triangles in \mathcal{T} with s being \mathcal{X} -monic by Lemma 2.2. Thus by Lemma 2.8, we have the following commutative diagrams

$$\begin{array}{ccccccc}
\bar{A} & \xrightarrow{\bar{f}} & \bar{B} & \xrightarrow{\bar{g}} & \bar{C} & \xrightarrow{\bar{h}} & \Sigma \bar{A} \\
\parallel & & \downarrow \bar{a} & & \downarrow \bar{s} & & \parallel \\
\bar{A} & \xrightarrow{\bar{a}\bar{f}} & \bar{X} & \xrightarrow{\bar{d}} & \bar{Z} & \xrightarrow{\bar{e}} & \Sigma \bar{A} \\
& & \downarrow \bar{b} & & \downarrow \bar{t} & & \\
& & \bar{Y} & \xlongequal{\quad} & \bar{Y} & & \\
& & \downarrow \bar{c} & & \downarrow & & \\
& & \Sigma \bar{B} & \longrightarrow & \Sigma \bar{C} & &
\end{array}$$

$$\begin{array}{ccccccc}
\bar{A} & \xrightarrow{\bar{a}\bar{f}} & \bar{X} & \xrightarrow{\bar{d}} & \bar{Z} & \xrightarrow{\bar{e}} & \Sigma \bar{A} \\
\downarrow \bar{f} & & \parallel & & \downarrow \bar{t} & & \downarrow \Sigma \bar{f} \\
\bar{B} & \xrightarrow{\bar{a}} & \bar{X} & \xrightarrow{\bar{b}} & \bar{Y} & \xrightarrow{\bar{c}} & \Sigma \bar{B}
\end{array}$$

of right triangles in $\mathcal{A}/[\mathcal{X}]$.

(2) Dual to (1). Let \triangle be the set of the left distinguished triangles of $\mathcal{A}/[\mathcal{Y}]$. We only need to prove the set \triangle is a left triangulation of $\mathcal{A}/[\mathcal{Y}]$. \square

Corollary 2.10. *Let \mathcal{A} be a full subcategory of a triangulated category \mathcal{T} , \mathcal{X} a functorially finite subcategory of \mathcal{A} . If \mathcal{A} has the properties that any \mathcal{X} -monomorphism of \mathcal{A} has a cone and any \mathcal{X} -epimorphism has a cocone. Then the subfactor category $\mathcal{A}/[\mathcal{X}]$ admits a pretriangulated structure in the sense of [BR].*

Proof. An easy modification of results of [BR, B] in our setting. \square

Remark 2.11. A pretriangulation $(\Omega, \Sigma, \Delta, \nabla)$ of \mathcal{C} becomes a triangulation if and only if $\Delta = \nabla$ and $\Omega = \Sigma^{-1}$.

Example 2.12. Let \mathcal{T} be a triangulated category and $\mathcal{X} \subset \mathcal{A}$ be subcategories of \mathcal{T} . If \mathcal{A} and \mathcal{X} satisfy the conditions of [IY]:

- (Z1) \mathcal{A} is extension closed,
- (Z2) $(\mathcal{A}, \mathcal{A})$ forms an \mathcal{X} -mutation pair.

Then by the definition of \mathcal{X} -mutation in [IY], \mathcal{X} is a functorially finite subcategory of \mathcal{A} and \mathcal{A} has the properties that any \mathcal{X} -monomorphism has a cone and any \mathcal{X} -epimorphism has a cocone by Lemma 4.3.(2) in [IY] and its dual. Moreover, the suspension functor coincides with the loop functor and is an auto-equivalence of $\mathcal{A}/[\mathcal{X}]$ [see Proposition 2.6 [IY]. Hence the subfactor category $\mathcal{A}/[\mathcal{X}]$ forms a triangulated category, that is the Theorem 4.2 of [IY].

In addition, we can prove the converse is also true.

Theorem 2.13. Let \mathcal{T} be a triangulated category and $\mathcal{X} \subset \mathcal{A}$ be subcategories of \mathcal{T} . \mathcal{A} and \mathcal{X} satisfy the conditions (Z1) and (Z2) if and only if the subfactor category $\mathcal{A}/[\mathcal{X}]$ forms a triangulated category where \mathcal{X} is a functorially finite subcategory of \mathcal{A} satisfying $(\mathcal{X}, \mathcal{X}[1]) = 0$ and \mathcal{A} has the properties that any \mathcal{X} -monomorphism has a cone and any \mathcal{X} -epimorphism has a cocone.

Proof. “ \Rightarrow ” Example 2.12.

“ \Leftarrow ” First of all, we claim $(\mathcal{X}, \mathcal{A}[1]) = 0$. Indeed, since the pretriangulated category $\mathcal{A}/[\mathcal{X}]$ forms a triangulated category, then $\Delta = \nabla$ and $\Omega = \Sigma^{-1}$ is an auto-equivalence of $\mathcal{A}/[\mathcal{X}]$. For any object A in \mathcal{A} , we have the following commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & X_A & \longrightarrow & \Sigma A & \longrightarrow & A[1] \\ \downarrow & & \downarrow & & \parallel & & \downarrow \\ \Omega \Sigma A & \longrightarrow & X_{\Sigma A} & \xrightarrow{\beta} & \Sigma A & \longrightarrow & \Omega \Sigma A[1] \end{array}$$

of triangles with α a left \mathcal{X} -approximation of A and β a right \mathcal{X} -approximation of ΣA . Since β is a right \mathcal{X} -approximation and $(\mathcal{X}, \mathcal{X}[1]) = 0$, we have $(\mathcal{X}, \Omega \Sigma A[1]) = 0$. Then by $\Omega \Sigma \bar{A} \cong \bar{A}$, i.e. $\Omega \Sigma A \oplus X' = A \oplus X''$, we get $(\mathcal{X}, A[1]) = 0$. Hence $(\mathcal{X}, \mathcal{A}[1]) = 0$ holds. Dually we can get $(\mathcal{A}, \mathcal{X}[1]) = 0$.

Let $A_2 \longrightarrow B \longrightarrow A_1 \longrightarrow A_2[1]$ be a triangle of \mathcal{T} with $A_1, A_2 \in \mathcal{A}$. Noting that $(\mathcal{X}, \mathcal{A}[1]) = 0$, we have the following diagram

$$\begin{array}{ccccccc} & & A_3 & \xlongequal{\quad} & A_3 & & \\ & & \downarrow s & & \downarrow \beta & & \\ A_2 & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & A_2 \oplus X & \xrightarrow{(0,1)} & X & \xrightarrow{0} & A_2[1] \\ \parallel & & \downarrow t & & \downarrow \alpha & & \parallel \\ A_2 & \longrightarrow & B & \longrightarrow & A_1 & \longrightarrow & A_2[1] \end{array}$$

of triangles in \mathcal{T} where α is a right \mathcal{X} -approximation of A_1 . Easy computation allow us to have $s = \binom{l}{\beta}$ for some l . Since α is a right \mathcal{X} -approximation and $(\mathcal{A}, \mathcal{X}[1]) = 0$, we have $A_3 \in \mathcal{A}$ and β is a left \mathcal{X} -approximation of A_3 . By Lemma 2.2, $s = \binom{l}{\beta}$ is \mathcal{X} -monic with $A_3, A_2, X \in \mathcal{A}$, hence $B \in \mathcal{A}$, i.e. \mathcal{A} is closed under extensions.

Since Σ is an auto-equivalence of $\mathcal{A}/[\mathcal{X}]$, for $\forall C \in \mathcal{A}$, there exists an object A such that $\Sigma\bar{A} = \bar{C}$, i.e. satisfying the following triangle

$$A \xrightarrow{\alpha_A} X_A \longrightarrow \Sigma A \longrightarrow A[1]$$

with α_A a \mathcal{X} -approximation of A and $X_A \in \mathcal{X}$. Then $C \oplus X_1 = \Sigma A \oplus X_2$ holds for some $X_1, X_2 \in \mathcal{X}$. Since $\Sigma A \oplus X_2 \in \mu^{-1}(\mathcal{A}; \mathcal{X})$, $(\mathcal{X}, \mathcal{A}[1]) = 0$, then $\mu^{-1}(\mathcal{A}; \mathcal{X}) = (\mathcal{X} * \mathcal{A}[1]) \cap^\perp \mathcal{X}[1]$ is closed under direct summands by Proposition 2.1 of [IY]. Thus $C \in \mu^{-1}(\mathcal{A}; \mathcal{X})$, i.e. $\mathcal{X} \subset \mathcal{A} \subset \mu^{-1}(\mathcal{A}; \mathcal{X})$.

Dually we can also get $\mathcal{X} \subset \mathcal{A} \subset \mu(\mathcal{A}; \mathcal{X})$. Hence $(\mathcal{A}, \mathcal{A})$ forms a \mathcal{X} -mutation pair. \square

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